

# On the Conformal Field Theory of the Higgs Branch

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**ABSTRACT:** We study 1 + 1-dimensional theories of vector and hypermultiplets with  $(4, 4)$  supersymmetry. Despite strong infrared fluctuations, these theories flow in general to distinct conformal field theories on the Coulomb and Higgs branches. In some cases there may be a quantum Higgs theory even when there is no classical Higgs branch. The Higgs branches of certain such theories provide a framework for a matrix model of Type IIA fivebranes and the associated exotic six-dimensional string theories. Proposals concerning the interactions of these string theories are evaluated.

**KEYWORDS:** Conformal Field Models in String Theory, M(atr ix) Theories.

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## 1 Introduction

In the matrix model of  $M$  theory [1], there are a number of problems in which one must study supersymmetric theories with eight supercharges that can arise by dimensional reduction from six dimensions. Such problems include  $M$  theory with longitudinal five-branes [2] or in the presence of an ALE singularity [3, 4]. To study these problems in uncompactified  $M$  theory, one must consider the dimensional reduction to 0+1 dimensions of a six-dimensional theory with vector and hypermultiplets.

Alternatively, instead of studying fivebranes or an ALE singularity in  $M$  theory, one can compactify on a circle and consider the same problems in weakly coupled Type IIA superstring theory. To do so, one must generalize matrix string theory [5, 6, 7] of Type IIA superstrings to include hypermultiplets as well as vector multiplets in the 1 + 1-dimensional theory.

For closely related reasons, the same sort of 1 + 1-dimensional theory of vector and hypermultiplets governs the behavior of a D-onebrane parallel to a D-fivebrane. Investigations in that context have shown [8] that the interaction with the hypermultiplet induces on the Coulomb branch (that is, on the vector multiplet moduli space) a metric with a rather peculiar behavior – a semi-infinite tube that appears at the “origin” in field space. The metric with the semi-infinite tube is in fact the familiar, but enigmatic, metric transverse to a fivebrane [9, 10]. 1 + 1-dimensional systems with several vector and hypermultiplets give generalizations of the tube metric such as an  $A - D - E$  family [11].

The Coulomb and Higgs branches of 1 + 1-dimensional theories of vector and hypermultiplets are distinct quantum mechanically [12] despite strong 1 + 1-dimensional infrared fluctuations. The main goal of section 2 is to try to interpret the tube metric on the Coulomb branch of vacua. The tube metric requires some explanation since the physics at the origin is surely non-singular. The interpretation that will be proposed here is that the tube metric on the Coulomb branch is a manifestation of a conformal field theory to which the superrenormalizable theory of vector and hypermultiplets can flow in the infrared. This conformal field theory is part of the Higgs branch if a classical Higgs branch exists; otherwise it is a sort of quantum Higgs branch (at the “origin”) that exists even in the absence of a classical Higgs branch.

In section three we apply the results to matrix theory. The main goal is to explore the models relevant to “matrix string theory” of Type IIA fivebranes. First we specialize to the particular cases of vector and hypermultiplets that are relevant to the matrix theory of  $k$  parallel longitudinal Type IIA fivebranes. For the same reasons as in [5, 6, 7], the limit in which the Type IIA coupling vanishes is the infrared limit of the 1 + 1-dimensional theory. In this limit, the string coupling vanishes but interactions on the fivebrane worldvolume survive [13] giving a fascinating theory that is roughly a “six-dimensional non-critical string theory.” This six-dimensional theory corresponds to the Higgs branch of the 1 + 1-dimensional theory. We attempt to determine the extent to which a recent proposal [14] can be used to describe the interactions in this six-dimensional theory; the results are interesting but not entirely conclusive.

At an advanced stage of the present work, I learned of related work [15] on the same models and especially their 0 + 1-dimensional counterparts. Related ideas were also developed by M. Douglas, and I benefited from discussions with him.

## 2 The Coulomb Branch And The Higgs Branch

### 2.1 Kinematics

We begin with a six-dimensional supersymmetric theory with a gauge group  $G$  and hypermultiplets transforming in the representation  $R$  of  $G$ . We write  $n_V$  and  $n_H$  for the number of vector multiplets and hypermultiplets (the dimension of  $G$  and the quaternionic dimension of  $R$ , respectively) and  $r$  for the rank of  $G$ . The six-dimensional theory has an  $SU(2)$   $R$  symmetry  $K$  which acts trivially on the gauge fields (and therefore non-trivially on their fermionic partners) while the bosonic fields in the hypermultiplets transform with  $K = 1/2$  (and their fermionic partners are invariant under  $K$ ).

Upon dimensional reduction to 1+1 dimensions, the six-dimensional vector and hypermultiplets reduce to two-dimensional vector and hypermultiplets.<sup>1</sup> The six-dimensional gauge field splits into a two-dimensional gauge field plus scalars  $\phi_i$ ,  $i = 1, \dots, 4$  in the adjoint representation of  $G$ . The dimensional reduction produces an extra  $SO(4)$   $R$

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<sup>1</sup>In two dimensions there are (4, 4) multiplets other than the vector and hypermultiplets [16], though they will not enter our discussion in the present paper.

symmetry group, under which the  $\phi_i$  transform as a vector and the scalars in the hypermultiplets are invariant. We write this  $SO(4)$  (or rather its double cover) as  $L \times L'$ , where  $L$  and  $L'$  are copies of  $SU(2)$ .

In the present section, all Fayet-Iliopoulos  $D$  terms and hypermultiplet bare masses are set to zero and we assume that no hypermultiplets are neutral under  $G$ .  $n_H$  free hypermultiplets would admit an action of a symmetry group  $Sp(n_H)$ , commuting with the  $R$  symmetry group  $K$ . We write  $H^{AX}$  for the scalars in the hypermultiplets; here  $A = 1, 2$  is a  $K$  index (reflecting the fact that the  $H$ 's transform with  $K = 1/2$ ), and  $X = 1, \dots, 2n_H$  is an  $Sp(n_H)$  index. The gauge group  $G$  is (via the representation  $R$ ) a subgroup of  $Sp(n_H)$ . The  $a^{\text{th}}$  generator of  $G$ , for  $a = 1, \dots, n_V$ , is represented by a tensor  $t_{XY}^a$ , and we write  $\phi_i^a$ ,  $a = 1, \dots, n_V$  for the components of  $\phi_i$ .

At the classical level, a supersymmetric state is simply a zero of the potential energy  $V$  of the model.  $V$  is a sum of three terms which are multiples of

$$\begin{aligned}
 V_1 &= \sum_{i < j} \text{Tr}[\phi_i, \phi_j]^2 \\
 V_2 &= \sum_{A, X, i} \left| \sum_{a, Y} \phi_i^a t_{XY}^a H^{AY} \right|^2 \\
 V_3 &= \sum_{A, B, a} D^{ABa} D_{AB}^a.
 \end{aligned}
 \tag{2.1}$$

In the third line,  $D^{ABa} = t_{XY}^a H^{AX} H^{BY}$  is the hyper-Kähler moment map associated with the action of  $G$  on the  $H$ 's, preserving the hyper-Kähler structure.

At a zero of  $V$ , all three terms separately vanish. The vanishing of  $V_1$  implies that the  $\phi_i$  commute. If the  $\phi_i$  commute and are otherwise generic, then vanishing of  $V_2$  implies that  $H = 0$ . So there is always a ‘‘Coulomb branch’’ of classical vacua in which the  $\phi_i$  are commuting and otherwise generic, and  $H = 0$ .

If the representation  $R$  is large enough, there is also a ‘‘Higgs branch’’ of vacua in which  $H \neq 0$  and  $\phi_i = 0$ . There may also be mixed Coulomb-Higgs branches in which the  $\phi_i$  are non-zero but not generic, and some hypermultiplets are non-zero. However, in this paper we consider only the Coulomb and Higgs branches.

## 2.2 Separation Of The Two Branches

In a similar situation above 1 + 1 dimensions, the classical Coulomb and Higgs branches just described are first approximations to quantum Coulomb and Higgs branches, parametrizing families of quantum vacua. In 1 + 1 dimensions, because of strong infrared fluctuations of massless scalars, one does not usually encounter continuous moduli spaces of quantum vacua. Roughly speaking, the ground state wave function on the Coulomb branch spreads over the whole Coulomb branch, and likewise the ground state wave function on the Higgs branch spreads over the whole Higgs branch. (In most instances the branch in question is non-compact and there is actually no normalizable quantum ground state wave function at all, but in any event the quantum mechanical states spread over the Coulomb or Higgs branches.)

Since the Coulomb and Higgs branches meet at  $\phi = H = 0$ , one may question whether the Coulomb and Higgs branches are actually distinct quantum mechanically. May not the wave function spread from the Coulomb to the Higgs branch? However, it has been argued [12] that these branches really are distinct even in the quantum theory. This follows upon considering the (4, 4) superconformal field theories that the (4, 4) supersymmetric gauge theory flows to in the infrared.

A (4,4) superconformal field theory has an  $SU(2)_L \times SU(2)_R$  group of left-moving and right-moving  $R$  symmetries. Conformal symmetry implies that the  $R$  symmetry currents must be purely left-moving or purely right-moving, and hence cannot rotate scalar fields (a symmetry that rotates scalar fields cannot be separated locally into left- and right-moving pieces).<sup>2</sup> It follows then that the Higgs and Coulomb branches have separate  $R$ -symmetries. The group  $L \times L'$  described at the beginning of this section acts trivially on the scalar fields  $H^{AX}$  that are in the hypermultiplets, and so can be the  $R$  symmetry group of a (4,4) superconformal field theory obtained by infrared flow on the Higgs branch. Since  $L$  and  $L'$  act non-trivially on the scalars  $\phi_i$  that parametrize the Coulomb branch, they cannot appear as  $R$  symmetries in a superconformal field theory derived from the Coulomb branch. The candidate  $R$  symmetry of the Coulomb branch, because it acts trivially on the  $\phi_i$ , is  $K$  (this is only one  $SU(2)$  symmetry; to get the expected  $SU(2)_R \times SU(2)_L$   $R$ -symmetry group of a (4, 4) model,  $K$  must break up in the infrared as the sum of separately conserved left- and right-moving symmetries). As the Higgs and Coulomb branches have different  $R$ -symmetries, they must in fact be different conformal field theories.

In most cases, we could alternatively reason as follows. The Coulomb branch is parametrized by  $r$  massless vector multiplets; all other fields are generically massive. So the superconformal field theory on the Coulomb branch has central charge  $\hat{c} = r$ .<sup>3</sup> On the other hand, if the representation  $R$  is big enough so that the gauge group is generically completely broken on the Higgs branch, then the Higgs branch is parametrized by  $n_H - n_V$  massless hypermultiplets with all other fields generically massive. So the superconformal field theory on the Higgs branch has central charge

$$\hat{c} = n_H - n_V. \quad (2.2)$$

In most examples,  $r \neq n_H - n_V$ , and this inequality of the central charges shows that the Coulomb and Higgs branches are different. (The example considered in [12] was an exception with  $r = n_H - n_V = 1$ .)

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<sup>2</sup>To state this argument very carefully, one should use the fact that generally the Coulomb and Higgs branches are noncompact and consider regions at infinity that can be analyzed semiclassically (all radii of curvature being large compared to the string scale). In such a region, a symmetry that rotates scalar fields cannot be locally separated into left- and right-moving pieces. In any model, therefore, in which the Coulomb or Higgs branches contain such semiclassical regions, their  $R$  symmetries cannot act on the massless scalar fields of the branch in question.

<sup>3</sup>In the usual examples of (4, 4) superconformal field theory, including all examples that will be considered in the present paper, the left- and right-moving central charges are equal; so we do not indicate a potential difference between them in the notation. The left- and right-moving Kac-Moody levels, introduced below, are likewise equal in these models.

Equation (2.2) can be justified more generally by considering the  $R$  symmetries and *without* assuming that the gauge group is completely broken on the Higgs branch. In (4,4) superconformal field theory,  $\hat{c}$  is equal to the “level” of the Kac-Moody algebra of the left or right-moving  $SU(2)$  group of  $R$  symmetries. This level can be computed even away from criticality, for it equals the anomaly in the two point function of the  $R$  symmetry currents. The symmetry  $L$ , for example, couples to the left-moving fermions in the hypermultiplets and the right-movers in the vector multiplets; so its anomaly is  $n_H - n_V$ ; and this is its Kac-Moody level in any possible conformal infrared limit of the Higgs branch. (Away from criticality,  $L$  couples to left and right-moving bosons in the vector multiplet; it becomes purely left-moving only in an infrared limit.) Hence a conformal field theory in which  $L$  is an  $R$ -symmetry has  $\hat{c} = n_H - n_V$ .

So we get a definition of a quantum Higgs branch that does not make any assumption about breaking of the gauge symmetry at the classical level. A (4,4) superconformal field theory to which the gauge theory can flow in the infrared is a conformal field theory of the Higgs branch if the  $R$  symmetry group is  $L \times L'$  and (therefore) the central charge is  $\hat{c} = n_H - n_V$ . Note that a quantum Higgs branch in this sense cannot exist if  $n_H - n_V < 0$ . If on the other hand  $n_H - n_V = 0$ , then the quantum Higgs branch has  $\hat{c} = 0$  and so is an infrared-trivial theory with a mass gap. Since any theory with such a  $\hat{c} = 0$  quantum Higgs vacuum also has a quantum Coulomb branch with the same minimum energy and no mass gap, such a quantum Higgs vacuum with mass gap is a sort of “bound state at threshold” of the quantum field theory. It is somewhat analogous to bound states at threshold whose existence follows in certain cases from string duality and can be proved via index theory [17]. It would be nice to know for sure if such quantum Higgs vacua exist in two-dimensional (4,4) theories; decisive arguments one way or the other will not be given in the present paper.

### 2.3 Interpretation Of The Tube Metric

Now we will focus on a specific case, which is  $U(1)$  gauge theory with  $k$  hypermultiplets.

At the classical level, the metric on the Coulomb branch of the  $U(1)$  gauge theory is simply the flat metric  $|d\phi|^2/2e^2$ . A one-loop diagram with internal hypermultiplets [8] induces on the Coulomb branch a quantum metric

$$ds^2 = |d\phi|^2 \left( \frac{1}{2e^2} + \frac{k}{2|\phi|^2} \right). \tag{2.3}$$

This metric is actually the same as the metric transverse to a Type IIA fivebrane [9, 10], a fact which is closely related to the fact that in matrix theory one can interpret [2] the inclusion of a hypermultiplet as the incorporation of a fivebrane in  $M$  theory.

The metric in (2.3) is enigmatic because there is a semi-infinite tube, isometric to  $\mathbf{R} \times \mathbf{S}^3$  with the obvious homogeneous metric, near  $\phi = 0$ . In particular,  $\phi = 0$  is “infinitely far away.” This seems strange because in the theory of vector and hypermultiplets, it does not seem intuitively that  $\phi = 0$  is at an infinite distance. Higher loop corrections to

(2.3) cannot resolve the enigma, since [8] supersymmetry ensures that the metric receives no further corrections.

It has been proposed [12] that the occurrence of such tube metrics is related to the fact that the Coulomb and Higgs branches separate under renormalization group flow. The intuitive idea was that  $\phi = 0$  (where the Coulomb and Higgs branches meet) is “infinitely far away” by the time one flows to the infrared. The precise form of the tube metric suggests, however, a further and more specific interpretation. The classical metric corresponds to a two-dimensional Lagrangian

$$\int d^2x \frac{|d\phi|^2}{2e^2} \tag{2.4}$$

that is conformally invariant if and only if  $\phi$  has dimension zero, the usual conformal dimension for a scalar field in two dimensions. But the tube metric is such that the Lagrangian near the origin

$$k \int d^2x \frac{|d\phi|^2}{2|\phi|^2} \tag{2.5}$$

is conformally invariant for any conformal dimension that might be assigned to  $\phi$ . This strongly suggests that what is happening at the origin is that there is a conformal field theory in which  $\phi$  has a non-zero conformal dimension. This is further suggested by the fact [9] that  $\ln|\phi|$  behaves as a Liouville field in the theory in the tube.

A simple inspection of (2.5) does not reveal the conformal dimension of  $\phi$ , but consideration of where (2.5) came from makes it clear what the dimension must be. (2.5) was obtained by integrating out the hypermultiplets from an action that is schematically

$$\sum_{\alpha=1}^k \int d^2x \left( |(d+A)H_\alpha|^2 + |\phi|^2 |H_\alpha|^2 + \bar{\psi}_\alpha \Gamma \cdot (d+A)\psi_\alpha + \phi \bar{\psi}_\alpha \psi_\alpha + \lambda H_\alpha \psi_\alpha + D H_\alpha H_\alpha \right). \tag{2.6}$$

Here  $H_\alpha$ ,  $\psi_\alpha$ ,  $\alpha = 1, \dots, k$  are the bosons and fermions in the  $\alpha^{th}$  hypermultiplet; the vector multiplet consists of the scalars  $\phi$ , gauge field  $A$ , fermions  $\lambda$ , and auxiliary field  $\vec{D}$ , whose interactions with the hypermultiplet are sketched in (2.6). The full Lagrangian contains in addition to the hypermultiplet action (2.6) the vector multiplet action which is schematically

$$\frac{1}{2e^2} \int d^2x \left( |F|^2 + |d\phi|^2 + \bar{\lambda} \Gamma \cdot \partial \lambda + \vec{D}^2 \right). \tag{2.7}$$

$F = dA$  is the electric field.

Now (2.6) is conformally invariant if and only if we assign dimension 1 to  $\phi$ , 3/2 to  $\lambda$ , 1 to  $A$ , and 2 to  $D$  and canonical dimensions (0 for  $H$ , 1/2 for  $\psi$ ) for the hypermultiplets. These dimensions for the vector multiplet are not the usual canonical dimensions, but they are “geometrical” dimensions; for instance, dimension one is the natural geometrical dimension for a gauge field (this is clear from the fact that  $A_i$  appears with  $\partial/\partial x^i$  in the covariant derivative  $D_i = \partial_i + A_i$ ), and the others are determined from this by supersymmetry.

Since (2.5) is obtained from (2.6) by expanding near constant  $\phi$  and near zero momentum and performing the path integral over the hypermultiplets, conformal invariance

of (2.5) is a consequence of conformal invariance of (2.6). If (and only if) we assign geometrical dimensions to the fields in the vector multiplet, the entire hypermultiplet path integral and not just the particular term extracted in (2.5) will be conformally invariant.

Once we assign dimensions 1, 3/2, 1, 2 to  $\phi, \lambda, A, D$ , the vector multiplet kinetic energy (2.7) is “irrelevant” (all terms have dimension four) and can be dropped in flowing to the infrared. Thus, we are dealing with “induced gauge theory” in which the vector multiplet kinetic energy comes completely from performing the path integral over the hypermultiplets. One might question to begin with whether such an induced gauge theory actually makes sense. At least for large  $k$ , one can argue as follows that it does. The path integral over the hypermultiplets induces a kinetic energy for the vector multiplets of the general form

$$kL_{eff}(\phi, \lambda, A, D). \quad (2.8)$$

The overall factor of  $k$  means that the effective theory of the vector multiplet is weakly coupled for large  $k$ . The expression  $L_{eff}$  is conformally invariant and has a non-degenerate expansion around  $\phi = \lambda = A = D = 0$ , though – as one would expect in an interacting conformal field theory – its expansion near zero momentum is rather delicate. The expansion in powers of  $1/k$  can be made systematically using propagators and vertices obtained by expanding  $L_{eff}$  in powers of the fields. No infinities arise in the perturbation expansion since there are no possible marginal or relevant (dimension  $\leq 2$ ) local gauge-invariant and supersymmetric counterterms in this theory. (The expansion in powers of  $1/k$  is somewhat analogous to the  $1/N$  expansion of the two-dimensional  $(\bar{\psi}\psi)^2$  model above two dimensions, which was originally suggested in [18].)

The conformal field theory obtained in this way is, if  $k$  is sufficiently large, to be attributed to the Higgs branch and not the Coulomb branch. For in this theory  $H$  has dimension zero and can obtain an expectation value without breaking conformal invariance, while  $\phi$  has dimension one and cannot have an expectation value. (On the Coulomb branch, of course,  $\phi$  has dimension zero and can have an expectation value.) The central charge of this conformal field theory is  $\hat{c} = k - 1$  according to (2.2). The first term in the  $1/k$  expansion would give the leading approximation  $\hat{c} \cong k$ .

Thus, for large  $k$  the conformal field theory obtained in this way is not really “new,” but gives a useful way to describe the behavior of the Higgs branch near the origin. Since the basic puzzle of interpreting the tube metric of the Coulomb branch exists for all  $k \geq 2$  [11], it is natural to propose that this description is valid all the way down to and including  $k = 2$ . It seems that the puzzle of the tube metric may not be present for  $k = 1$  [11], and this suggests that for  $k = 1$  a quantum Higgs vacuum may not be present. Note that for  $k = 1$ , a Higgs vacuum would have  $\hat{c} = 0$  and so would correspond to a theory with mass gap that would be trivial (and not just free) in the infrared limit. Existence of a Higgs vacuum for  $k = 1$  would be important in the theory of fivebranes, as will be clear in section 3, so it would be desirable to obtain decisive arguments showing that such a vacuum does or does not exist.



## 2.4 Fixed Points in Superrenormalizable Gauge Theories

We will try to put this discussion on somewhat firmer ground by observing that somewhat similar phenomena occur in many superrenormalizable gauge theories in two and three dimensions.

Consider a  $U(1)$  gauge theory with gauge field  $A$  and field strength  $F$ . Classically the object  $I(\Sigma) = \int_{\Sigma} F$  is a topological invariant, for any closed two-surface  $\Sigma$  in spacetime. One possible behavior of such a theory is that the gauge field  $A$  may decouple in the infrared limit. If so, then the different gauge bundles labeled by different values of  $I(\Sigma)$  are indistinguishable in the infrared and  $F$  can have any dimension at all. Usually what happens in superrenormalizable theories is that  $F$  has dimension less than 2 (for instance, dimension 1 on the Coulomb branch of the supersymmetric two-dimensional theories that we have been considering), in which case  $I(\Sigma)$  diverges as one flows to the infrared and is not a well-defined observable of the conformal field theory.

Alternatively, one may flow to an infrared fixed point at which  $A$  does not decouple from charged fields. In that case,  $F$  has dimension exactly 2, since this is the only value compatible with topological invariance of  $I(\Sigma)$ .

This discussion is probably not limited to abelian gauge theory. The intuitive idea is that gauge fields of any gauge group either decouple in the infrared from charged fields (including themselves in the non-abelian case) or have canonical dimensions. A more precise statement and argument is more difficult to give in the nonabelian case because a gauge-invariant operator as convenient as  $F$  does not exist.

We will illustrate these ideas with a number of examples.

**QED In Two Dimensions** First we consider two-dimensional QED, that is a  $U(1)$  gauge theory in two dimensions coupled to  $k$  complex fermions  $\psi_{\alpha}$ ,  $\alpha = 1, \dots, k$  of the same charge. The Lagrangian of this completely nonsupersymmetric model is

$$L = \frac{1}{4e^2} \int d^2x F_{ij} F^{ij} + \sum_{\alpha} \int d^2x i\bar{\psi}_{\alpha} \Gamma \cdot D\psi_{\alpha}. \tag{2.9}$$

Imitating the discussion of the supersymmetric theory, we are led to suspect that at least for sufficiently large  $k$ , this theory flows in the infrared to a conformal field theory that can be obtained by throwing away the gauge kinetic energy and integrating out the fermions to induce an effective kinetic energy.

This is in fact so for all  $k$ , as can be seen by bosonization. One can replace the fermions by bosons  $\phi_{\alpha}$ ,  $\alpha = 1, \dots, k$ , and the Lagrangian by<sup>4</sup>

$$L' = \frac{1}{4e^2} \int d^2x F_{ij} F^{ij} + \sum_{\alpha=1}^k \int d^2x \left( \frac{1}{2} |d\phi_{\alpha}|^2 + \frac{\phi_{\alpha} \epsilon^{ij} F_{ij}}{\sqrt{\pi}} \right). \tag{2.10}$$

This flows as expected to a conformal field theory in which  $F$  has dimension 2,  $\phi$  has dimension 0, and the original  $F^2$  kinetic energy is irrelevant in the infrared and can simply be dropped.

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<sup>4</sup>See [19] for this procedure and generalizations to analyze fermion mass terms.

The central charge can be computed using the fact that one combination of the scalars (namely their sum) combines with  $F$  to a massive field (the mass is infinite if one discards the  $F^2$  term from the Lagrangian), while the other  $k - 1$  scalars are free and massless. So in fact  $c = k - 1$ . At  $k = 1$ , since  $c = 0$ , we have a theory with a mass gap. In fact, this theory is the Schwinger model, long known to be equivalent by the above procedure to the theory of a massive free boson. The massive theory at  $k = 1$  is analogous to the massive Higgs vacuum that might possibly exist for the supersymmetric  $U(1)$  theory with one hypermultiplet, except that (as there is no Coulomb branch) it is not embedded in a continuum.

One oddity of the present model is that, while the correlation functions are conformally invariant in the infrared if  $F$  is assigned dimension two, connected correlation functions involving  $F$  have no infrared singularities. This would not be so in the supersymmetric example considered earlier if  $k > 1$ ; it is true, of course, for  $k = 1$  as there is a mass gap and no correlation functions have infrared singularities.

The  $1/k$  expansion has a similar qualitative structure in the nonabelian case. For this, one replaces the gauge group by a nonabelian group and takes the fermions to consist of  $k$  copies of any given representation. The Lagrangian is formally just like (2.9) (except that of course the gauge kinetic energy is now  $\text{Tr}F^2$ ) and there is a large  $k$  expansion as in the abelian case. For a recent brief discussion see [20]. One can even extend the discussion to small  $k$  via the relation between gauged WZW models and coset conformal field theories [21, 22, 23, 24, 24]. To do so, let  $k'$  be the total number of real fermi fields in the model, summed over representations. One can replace the fermions by an  $SO(k')$  WZW model at level one. Upon deleting the gauge kinetic energy from the Lagrangian, one gets a gauged WZW model which flows in the infrared to a coset conformal field theory. Using the machinery of coset conformal field theory, one can argue that the gauge fields have their canonical dimension in this infrared fixed point and one can describe the fixed point quantitatively.

**QED In Three Dimensions** Next we consider the same model –  $U(1)$  coupled to  $k$  species of complex fermion – but now in three dimensions. The  $1/k$  expansion of this theory has been extensively studied over a period of many years, some of the references being [25, 26, 27, 28, 29, 30]. In particular, it is demonstrated that order by order in  $1/k$ , there is a non-trivial infrared fixed point at which  $F$  has dimension two and the ordinary kinetic energy is irrelevant. The fact that the dimension of  $F$  is precisely two to all orders in  $1/k$  is argued in [30]. A difference from the two-dimensional case is that it has been argued [29, 30] that the flow to a non-trivial infrared fixed point of this nature occurs only for  $k$  greater than a certain critical value.

As in two dimensions, one can generalize to the nonabelian case without changing the qualitative structure of the  $1/k$  expansion. In fact, the argument in [30] showing dimension 2 for  $F$  is made in this context.

**Supersymmetric QED In Three Dimensions** For our last such example, we consider supersymmetric QED in four dimensions. For brevity we consider only the case of

$N = 4$  supersymmetry (that is, eight supercharges) though one could likewise consider  $N = 2$  supersymmetry [31] to give further examples.

So we consider a three-dimensional  $U(1)$  gauge theory of a vector multiplet coupled to  $k$  hypermultiplets. On the Coulomb branch there is a gauge field  $A$  and three scalars  $\vec{\phi}$ . After dualizing the gauge field to a fourth scalar, the Coulomb branch is described by a Taub-NUT hyper-Kähler metric with an  $A_{k-1}$  orbifold singularity [32, 33]. The Taub-NUT metric corresponds in terms of the original variables  $\vec{\phi}$ ,  $A$ , to a kinetic energy which looks like

$$k \int d^3x \left( \frac{|d\vec{\phi}|^2 + F^2}{|\vec{\phi}|} \right) \tag{2.11}$$

(with just such coefficients as to produce an  $A_{k-1}$  orbifold singularity after duality). We see the familiar fact that conformal invariance holds if and only if the fields are assigned their geometrical dimensions, namely 2 for  $F$  and 1 for  $\vec{\phi}$ . These are the dimensions that make the hypermultiplet kinetic energy conformally invariant, and if one did not have exact methods to analyze this problem, one could arrive at (2.11) as the behavior near the origin for large  $k$  by simply integrating out the hypermultiplets. Since there are exact methods, one can be precise about how large  $k$  must be to produce such behavior at the origin. The form (2.11) for the metric near  $\vec{\phi} = 0$  is valid for all  $k \geq 1$ . For  $k \geq 2$ , this behavior at the origin actually corresponds to a critical point with non-trivial infrared behavior, while for  $k = 1$ , a duality transformation (to the scalar field dual to  $A$ ) eliminates the singularity, giving a smooth Taub-NUT metric. So in that case, the conformal field theory at the origin is actually infrared-free (but non-trivial, that is there is no mass gap) if expressed in the right variables.

**QED In Four Dimensions** We conclude by considering a model that is *not* super-renormalizable, namely QED in four dimensions. In this case, the fact that  $F$  must have dimension two at a critical point at which the photon does not decouple from electrons implies [34] that  $U(1)$  gauge theory in four dimensions cannot have such an infrared fixed point. This argument holds only in the absence of magnetic monopoles, since the Bianchi identity, which is used to show that  $I(\Sigma)$  is topologically invariant and  $F$  must have dimension 2, holds only in the absence of monopoles.

### 3 Matrix Theory With Hypermultiplets

#### 3.1 Models

In this section, we consider matrix string theory with hypermultiplets.

To incorporate longitudinal fivebranes we follow [2]. Type IIA matrix string theory is based on a  $U(n)$  gauge theory with  $(8, 8)$  supersymmetry. From the point of view of  $(4, 4)$  supersymmetry, this model has a  $U(n)$  vector multiplet and a hypermultiplet, which we will call  $X$ , in the adjoint representation of  $U(n)$ . To incorporate  $k$  parallel longitudinal fivebranes, one must add  $k$  hypermultiplets  $H_\alpha$ ,  $\alpha = 1, \dots, k$  in the fundamental

representation of  $U(n)$ .<sup>5</sup> Lorentz-invariant physics hopefully emerges in the large  $n$  limit.

Alternatively, to describe matrix string theory in the presence of an  $A - D - E$  singularity, one considers [3, 4] a certain collection of unitary groups and hypermultiplets associated with the  $A - D - E$  Dynkin diagram.

In the  $A - D - E$  case, the Fayet-Iliopoulos or FI couplings are quite important and lead to resolution of the singularity. In the presence of generic FI terms, the model has a Higgs branch and no Coulomb branch. The Higgs branch is interpreted physically in terms of motion of supergravitons in spacetime. The physical interpretation of the Coulomb branch that occurs when the FI terms vanish is an intriguing question that will not be considered here. We will focus instead on fivebranes.

In studying fivebranes, the goal is to understand the intrinsic fivebrane theory obtained [13] by taking the Type IIA string coupling constant to zero, whereupon a residual six-dimensional theory survives. Upon a further reduction ( $\alpha' \rightarrow 0$ ) this theory reduces to the exotic six-dimensional field theory found earlier [12, 36]. As in conventional Type IIA matrix string theory [5, 6, 7], the weak coupling limit is obtained by formulating the 1+1-dimensional theory on a circle  $S$  of radius  $R$ , and taking  $R \rightarrow \infty$ . The large  $R$  behavior is controlled by the possible conformal field theories to which the 1+1-dimensional gauge theory can flow.

An important point is that an intrinsic interacting fivebrane theory in the limit that the Type IIA string coupling constant goes to zero is only known to exist for the case of  $k \geq 2$  parallel fivebranes. The limiting theory must be non-trivial in this case since even in the infrared limit it flows to the non-trivial theory described in [12, 36]. In the analogous Type IIB case the argument that the fivebrane theory remains interacting as the string coupling constant goes to zero [13] uses the fact that the gauge theory on the fivebrane world-volume has a  $U(k)$  gauge group, which is non-abelian for  $k \geq 2$ . Thus, one only has a firm argument that an intrinsic fivebrane theory exists in the case  $k \geq 2$ . Because this theory seems to be the  $A_{k-1}$  case of an  $A - D - E$  story, and  $A_{k-1}$  corresponds to  $SU(k)$  rather than  $U(k)$ , one might suspect that the  $U(1)$  is completely decoupled (and not just decoupled in the infrared limit) and that there is no intrinsic theory in this sense for a single fivebrane.

The  $U(n)$  theory with an adjoint and  $k$  fundamental hypermultiplets  $H_\alpha$ ,  $\alpha = 1, \dots, k$ , has both a Coulomb branch and a Higgs branch. (The classical Higgs branch has rather special properties for  $k = 1$ , as we discuss later, and is generic for  $k > 1$ .) As is clear from the metric (2.3), the motion on the Coulomb branch describes the motion perpendicular to the fivebrane. In fact, near infinity on the Coulomb branch, the  $H_\alpha$  can be neglected (their masses become large), and one reduces to Type IIA matrix string theory in the absence of fivebranes.

So the fivebranes must be described by the conformal field theory of the Higgs branch.

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<sup>5</sup>What is considered in [2] is the problem of adding a hypermultiplet to zerobrane quantum mechanics to represent an  $M$  theory fivebrane. To make a Type IIA construction, one considers, in a standard fashion, a periodic array of fivebranes. After making a sort of Fourier transform or  $T$ -duality transformation [35], one arrives at the 1+1-dimensional field theory with hypermultiplets representing fivebranes.

On this branch, the scalars in the  $U(n)$  vector multiplet are near zero, and the transverse space of the light cone matrix string theory is reduced from eight to four dimensions, the correct number to describe a six-dimensional theory in light cone gauge.

Whenever there exists an intrinsic fivebrane theory that survives for vanishing string coupling, the conformal field theory of the Coulomb branch must have some pathology. For if the Coulomb branch is governed by a completely well-behaved  $(4, 4)$  superconformal field theory (with no problems except the standard noncompactness at infinity), then this conformal field theory is the starting point for a systematic Type IIA perturbation expansion. It would surely be inconsistent to add fivebranes by hand to such a perturbation expansion. So (a) if the Coulomb branch conformal field theory is completely well-defined, it must give a completely self-contained description of physics in the field of the fivebrane and hence the usual modes propagating on the fivebrane world-volume must be described by ordinary vertex operators in this theory (as was assumed in the earliest work on fivebrane conformal field theory [9]), and (b) if there is a limiting non-trivial fivebrane theory in the limit of zero string coupling constant (as we expect at least for  $k \geq 2$ ) the Coulomb branch conformal field theory cannot be completely well-defined.

We independently expect a problem with the Coulomb branch for  $k \geq 2$  because of the tube metric (which apparently exists for  $k \geq 2$  [11]) whose existence leads to a failure of normalizability of the Coulomb branch vacuum near the fivebrane location (near  $\phi = 0$  in the language of section two) and a blow-up of the effective Type IIA string coupling constant. The problem with the Coulomb branch when there is an intrinsic fivebrane theory is analogous to the fact that one cannot expect to give a completely consistent description of the physics outside a black hole without including black hole degrees of freedom.

For  $k > 1$ , after specifying the orientation of an  $M$  theory fivebrane, five additional real numbers are needed to specify its transverse position. In [2], these parameters were interpreted as bare masses of the  $H_\alpha$ , or more exactly as relative bare masses. To be more precise, in the  $0 + 1$ -dimensional construction in [2], the  $\alpha^{th}$  hypermultiplet interacts not with the scalars  $\vec{\phi}$  in the vector multiplet but with  $\vec{\phi} + \vec{a}_\alpha$ , where (as  $\vec{\phi}$  has five components in  $0 + 1$  dimensions) each  $\vec{a}_\alpha$  is a five-component “transverse position vector” for the  $\alpha^{th}$  fivebrane. Addition of an overall constant to all  $\vec{a}_\alpha$  can be absorbed in adding a constant to  $\vec{\phi}$  (just as an overall constant in the positions of a collection of fivebranes can be absorbed in a translation).

In  $1 + 1$ -dimensions, the  $\vec{\phi}$  and  $\vec{a}_\alpha$  have only four components; the fifth component of the fivebrane position has a different interpretation that we discuss momentarily. The hypermultiplet bare masses  $\vec{a}_\alpha$  are relevant operators, of dimension one. If therefore we want these perturbations to survive and not dominate as  $R$  (the radius of the circle  $S$ ) becomes large, we must take the  $\vec{a}_\alpha$  to be of order  $1/R$  as  $R \rightarrow \infty$ .

What about the fifth component of the fivebrane position? After the  $T$ -duality to a  $1+1$ -dimensional model, this component becomes a Wilson line around  $S$ . The  $U(n)$  gauge field with which  $H_\alpha$  interacts can be supplemented by an  $\alpha$ -dependent additive constant, which is such that the holonomy around  $S$  contains an  $\alpha$ -dependent multiplicative factor

$e^{i\theta_\alpha}$ , where  $\theta_\alpha$  is the position of the fivebrane in the eleventh dimension – the dimension that was compactified to go from  $M$  theory to Type IIA superstrings. The  $\theta_\alpha$  should be held fixed as  $R \rightarrow \infty$ .

### 3.2 Analysis Of The Classical Higgs Branch

We will now analyze the classical structure of the Higgs branch  $\mathcal{M}_{n,k}$  of the  $U(n)$  theory with an adjoint and  $k$  fundamental hypermultiplets. The results will be useful later when we consider interactions.

The Higgs branch of this theory (with zero bare masses and FI terms) can be interpreted as the moduli space of  $n$ -instanton solutions of  $U(k)$  gauge theory on  $\mathbf{R}^4$ , partially compactified by incorporating small instantons.<sup>6</sup> This fact follows from the ADHM construction of instantons, and is important in many of the physical interpretations of the model. For  $k = 1$ , as  $U(1)$  is abelian, there are no “honest” instantons, and all solutions correspond to completely collapsed instantons. The completely collapsed instantons correspond to  $H = 0$ , so are represented by the values of the adjoint hypermultiplet only. We will momentarily give a direct proof, without using the relation to instantons, that  $H = 0$  in any vacuum of the  $k = 1$  theory.

To analyze the generic structure of the Higgs branch for any  $k$ , we follow the concluding portion of [37] and proceed as follows. Pick a complex structure on the four-dimensions of the light-cone fivebrane world-volume and view the  $(4, 4)$  model as a  $(2, 2)$  supersymmetric model. Under this reduced supersymmetry, the adjoint hypermultiplet  $X$  splits into a pair  $U, V$  of chiral multiplets in the adjoint representation. The  $H_\alpha$ ,  $\alpha = 1, \dots, k$ , become chiral multiplets  $A_\alpha, B_\alpha$  in the fundamental and antifundamental representations of  $U(n)$ . The vanishing of the  $(4, 4)$   $\vec{D}$  terms give a complex equation

$$[U, V]^i_j + \sum_\alpha A_\alpha^i B_{\alpha j} = 0 \tag{3.1}$$

together with a real equation which is the vanishing of the  $(2, 2)$   $D$  field, which we call  $D_{\mathbf{R}}$ . The Higgs branch is described by solving (3.1, setting  $D_{\mathbf{R}}$  to zero, and dividing by  $U(n)$ . If things are sufficiently generic (which in the present problem is true for  $k > 1$  but not for  $k = 1$ ), then setting  $D_{\mathbf{R}}$  to zero and dividing by  $U(n)$  is equivalent on a dense open set to dividing by the complexification  $GL(n, \mathbf{C})$  of  $U(n)$ .

Now we proceed to solve (3.1) and divide by  $GL(n, \mathbf{C})$ . Generically we can diagonalize  $U$ , say with eigenvalues  $u_i$ ,  $i = 1, \dots, n$ . Diagonalizing  $U$  fixes most of the  $GL(n, \mathbf{C})$  symmetry; what remains are the diagonal  $GL(n, \mathbf{C})$  transformations – forming a group  $(\mathbf{C}^*)^n$  – and the Weyl group – which acts by permutation of the eigenvalues. Equations (3.1) put no restrictions on the diagonal matrix elements  $v_i = V^i_i$ . The Higgs branch will thus be parametrized among other things by the values of the  $u_i$  and the  $v_i$ . One can

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<sup>6</sup>In this paper, whenever we speak of instanton moduli spaces on  $\mathbf{R}^4$ , we always have in mind based instantons; that is, in defining the moduli space one considers two instantons to be gauge-equivalent if and only if they are equivalent by a gauge transformation that equals the identity at infinity.

generically solve (3.1) for the off-diagonal  $V^i_j$  via

$$V^i_j = -\frac{A^i B_j}{u_i - u_j}, \quad \text{for } i \neq j. \quad (3.2)$$

Finally, vanishing of the diagonal terms in (3.1) puts a restriction on  $A$  and  $B$ , which is that

$$\sum_{\alpha} A_{\alpha}^i B_{\alpha i} = 0, \quad \text{for } i = 1, \dots, n, \quad (3.3)$$

with no sum over  $i$  in this equation.

The surviving data, for each fixed  $i$ , are the diagonal matrix elements  $u_i$  and  $v_i$ , together with the hypermultiplets  $A_{\alpha}^i, B_{\alpha i}$ .  $A_{\alpha}^i$  and  $B_{\alpha i}$  must obey (3.3) and one must impose an equivalence relation  $A \rightarrow \lambda A, B \rightarrow \lambda^{-1} B$  that comes from the residual  $\mathbf{C}^*$  symmetry. The data  $u, v, A, B$  with this condition and this equivalence relation make up a copy of the Higgs branch  $\mathcal{M}_{1,k}$  for  $n = 1$  with  $k$  hypermultiplets. (By the ADHM construction, this is the same as the one-instanton moduli space for the group  $U(k)$ .) Since we get  $n$  copies of this space, one for each  $i$ , and we then must divide by the Weyl group, which acts by permutations of the eigenvalues, the conclusion is that on a dense open set the desired Higgs moduli space  $\mathcal{M}_{n,k}$  is the symmetric product  $S^n \mathcal{M}_{1,k}$  of  $n$  copies of  $\mathcal{M}_{1,k}$ . Technically,  $\mathcal{M}_{n,k}$  is *birational* to  $S^n \mathcal{M}_{1,k}$ . Note that this identification of  $\mathcal{M}_{n,k}$  with  $S^n \mathcal{M}_{1,k}$  does not respect all of the symmetries of the problem; in fact, it depended on the choice at the outset of a complex structure.

Now we look more closely at the special case  $k = 1$ . In this case, there is no sum over  $\alpha$  in (3.3), which collapses to the condition that  $A^i B_i = 0$  for each  $i$ . So after a rearrangement of the eigenvalues, we can assume that  $A^i \neq 0, B_i = 0$ , for  $i = 1, \dots, n_1$ ,  $A^i = 0, B_i \neq 0$  for  $n_1 + i \leq i \leq n_2$ , and  $A^i = B_i = 0$  for  $i > n_2$ . In this basis, (3.2) shows that  $V^i_j$  is upper triangular. An examination of the equation  $D_{\mathbf{R}} = 0$  now shows that it can be satisfied only for  $A^i = B_i = 0$ . (Thus this is an exceptional case in which setting  $D_{\mathbf{R}} = 0$  and dividing by  $U(n)$  is not generically equivalent to dividing by  $GL(n, \mathbf{C})$ , because “the generic orbit is not stable.”) This confirms the claim made at the outset of the present discussion that for  $k = 1$ , the hypermultiplet  $H$  is identically zero on the Higgs branch. (3.1) now requires that  $U$  and  $V$  commute, so the Higgs branch is described by the eigenvalues  $u_i, v_i$ . (The vanishing of  $D_{\mathbf{R}}$  forbids solutions in which  $U$  and  $V$  have Jordan forms looking like  $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$  and cannot be diagonalized.) In particular, the moduli space  $\mathcal{M}_{1,1}$  is parametrized by a single pair of complex numbers  $u, v$ , and is a copy of  $\mathbf{R}^4$ . More generally,  $\mathcal{M}_{n,1}$  is parametrized by  $n$  pairs  $u_i, v_i$  up to permutation, and so is isomorphic to a symmetric product

$$\mathcal{M}_{n,1} = S^n \mathcal{M}_{1,1}. \quad (3.4)$$

There are a few differences between (3.4) and the corresponding and superficially similar relation between  $\mathcal{M}_{n,k}$  and  $\mathcal{M}_{1,k}$ :

1. The relation (3.4) was deduced from the exact statement that  $H = 0$  in any vacuum for  $k = 1$  and is true on the nose as a relation between the spaces involved; no blowup or birational transformation is involved. The fact that for general  $n$  the birational equivalence of  $\mathcal{M}_{n,k}$  with  $S^n \mathcal{M}_{1,k}$  involves the equation (3.2) which has a pole at  $u_i = u_j$  shows that in the latter case we are only getting a birational transformation, which breaks down when  $u_i = u_j$ .
2. Related to this, the hyper-Kähler metric on the moduli space  $\mathcal{M}_{n,1}$  is the obvious flat hyper-Kähler metric on the symmetric product  $S^n \mathbf{R}^4$ . This is deduced by setting  $A = B = 0$  and evaluating the classical Lagrangian for diagonal (but spacetime-dependent) matrices  $U, V$ . By contrast, since (3.2) shows that the off-diagonal matrix elements of  $V$  do *not* vanish for  $k > 1$ , the metric on  $\mathcal{M}_{n,k}$  does not have such an elementary relation to that on  $\mathcal{M}_{1,k}$  for  $k > 1$ .
3. Finally, (3.4) is a consequence of the fact that  $H = 0$  on the Higgs branch of the  $k = 1$  theory, which is a statement completely invariant under all symmetries of the problem, though our proof of it was not invariant. So the identification (3.4) is completely invariant. That is not so for the corresponding birational equivalence for  $k > 1$ , which depended on a choice of complex structure.

### 3.3 The Hamiltonian

Now we will try to make as explicit as possible the structure of the theory, first in the case  $k = 1$ .

We have seen in the last subsection that the Higgs branch for  $k = 1$  is parametrized by the adjoint hypermultiplet  $X$ , with  $H = 0$ .  $X$  contains four scalar fields  $X_\lambda$ ,  $\lambda = 1, \dots, 4$  in the adjoint representation of  $U(n)$ . A point on the Higgs branch is parametrized by  $X$ 's that mutually commute, modulo the action of  $U(n)$ . The simultaneous eigenvalues of the  $X_\lambda$  define a set of  $n$  points in  $\mathbf{R}^4$ , which are uniquely determined up to the action of the Weyl group, that is, up to permutation.

If one is far out on the Higgs branch, that is if the  $n$  points in  $\mathbf{R}^4$  are all far apart, then the gauge group  $U(n)$  is spontaneously broken to  $U(1)^n$ . At this point, the low energy theory looks like  $n$  copies of the  $n = k = 1$  system. This system consists of  $U(1)$  with a single charged hypermultiplet plus an adjoint-valued hypermultiplet, which, as  $U(1)$  is abelian, is simply free. It is possible that the  $U(1)$  theory with one charged hypermultiplet has no quantum Higgs vacuum and has a completely well-defined Coulomb branch conformal field theory. In this case, upon including also the free hypermultiplet from the adjoint, we get a theory which has only one quantum branch in which the scalars in the vector multiplet and the free hypermultiplet are both free to vary. This theory would describe both the physics outside the fivebrane and the fivebrane modes, so there would be no separate fivebrane theory that can be decoupled from the theory in bulk. This would have to be a completely well-defined conformal field theory (except for the usual subtleties coming from the noncompactness at infinity), and the  $k = 1$  fivebrane



modes would be described by ordinary vertex operators in this theory.

On the other hand, it is conceivable that  $U(1)$  with one charged hypermultiplet can flow in the infrared to a massive Higgs vacuum. If so, then the  $n = k = 1$  system has a quantum Higgs branch in which only the scalars in the free hypermultiplet can vary. This branch will lead to the existence of an intrinsic theory for a single Type IIA fivebrane.

We will proceed with the discussion of interactions assuming that such a branch exists for  $n = k = 1$ . Even if it turns out that this is not so, the discussion of this case will give useful practice for  $k > 1$  where there definitely is an intrinsic fivebrane theory.

So now we consider the case  $k = 1$ ,  $n \gg 1$ . Far out on the Higgs branch, the eigenvalues of the scalars in the adjoint hypermultiplet define  $n$  points in  $\mathbf{R}^4$  (and the scalars in the vector multiplet are locked near zero because of our assumptions about the  $n = k = 1$  system). The key question is what happens when some of the  $n$  points in  $\mathbf{R}^4$  come near by. We have determined that the Higgs branch  $\mathcal{M}_{n,1}$  is precisely the symmetric product  $Y_n = S^n \mathbf{R}^4$  of  $n$  copies of  $\mathbf{R}^4$ , with the obvious flat metric. At first sight this strongly suggests that the conformal field theory of the Higgs branch would be simply the conventional orbifold conformal field theory with target space  $Y$ . This assumption, however, leads to contradictions. If the conformal field theory of the Higgs branch is simply an orbifold obtained by dividing the symmetric product of  $n$  copies of *anything* by the symmetric group  $S_n$ , then the spectrum of the matrix string theory would be – by the standard logic of matrix string theory [5, 6, 7] – a free Fock space of one particle states. Moreover, in this case the one particle states in question would be, essentially as guessed in [38], the states of the six-dimensional light-cone Green-Schwarz superstring. But the Green-Schwarz superstring in six-dimensions is not Lorentz-invariant, six not being the critical dimension. So whatever the theory of a single fivebrane (at vanishing Type IIA string coupling constant) may be, it cannot have the spectrum of the light-cone Green-Schwarz superstring.

A resolution of this problem was proposed in [14]. The proposal (as adapted to the language of the present discussion) is that the Higgs branch of the  $U(n)$  theory with  $k = 1$  is not precisely the orbifold conformal field theory but is a sigma model whose target space is a blow-up of  $Y$ . The idea, to be more precise, is that the locus that is blown up is the locus in  $Y$  in which *two* of the  $n$  points in  $\mathbf{R}^4$  coincide. This locus is of codimension four, and its blowup is represented by marginal deformations of the conformal field theory. Blowup of “higher order diagonals” on which more than two points coincide would by contrast be represented by irrelevant operators. Since we are blowing up a locus where just two points coincide, the singularity that is being blown up is a  $\mathbf{Z}_2$  or  $A_1$  orbifold singularity; the local behavior of the conformal field theory near such a singularity can in fact be described by a  $U(1)$  theory with two charged hypermultiplets (of equal charge), as in [12]. Geometrically, the blowup creates a two-cycle  $C$  (which is topologically a two-sphere).

The proposal to describe the fivebrane via a sigma model whose target space is this blowup cannot be precisely correct. The blowup of  $Y$  would violate the rotation symmetry of  $\mathbf{R}^4$  (breaking  $SO(4)$  to  $SU(2) \times U(1)$ ); but this  $SO(4)$ , which rotates the four transverse

dimensions of the fivebrane, is an exact symmetry of the fivebrane theory. Even more specifically, we saw in the last subsection that the Higgs branch of the  $U(n)$  theory with  $k = 1$  is equal on the nose to the symmetric product  $Y$ , and is not a blowup of  $Y$ .

However, a variant of the proposal in [14] does seem to be viable. The three blowup modes of the orbifold conformal field theory associated with blowup of the codimension four singular locus in  $Y$  have a supersymmetric completion which is a fourth marginal operator (like the blowup modes this operator is a twist field of the orbifold theory). The coupling constant associated with this operator would be interpreted after blowup as a world-sheet theta angle; before the blowup, it is hard to interpret this coupling as a worldsheet theta angle (since the two-cycle  $C$  is collapsed at the orbifold fixed point), but it can be interpreted as in [12] as the theta angle of a  $U(1)$  gauge theory that is used in an effective description near the  $A_1$  orbifold singularity.

A deformation of the theta angle does not violate the  $SO(4)$  transverse rotation symmetry of the fivebrane, but a generic such deformation does violate the left-right symmetry on the worldsheet of the  $1 + 1$ -dimensional theory. There are two values of  $\theta$ , namely  $\theta = 0$  and  $\theta = \pi$ , which do respect this left-right symmetry and in fact respect all symmetries that should be manifest for the light cone fivebrane. An important fact in this subject is that the conventional orbifold theory – the one that can be expressed in terms of free fields – is the theory at  $\theta = \pi$  [39]. The fivebrane theory cannot be described by the conventional orbifold, for reasons already explained, so  $\theta = \pi$  is ruled out. We are thus left with one option along these lines: the conformal field theory on the  $k = 1$  Higgs branch is not the usual orbifold, but differs from it by having  $\theta = 0$  instead of  $\theta = \pi$ . It seems that this is the appropriate version of the conjecture made in [14].

As encouraging hints supporting this interpretation, we may note the following:

1. There is a unique and fairly elegant candidate, an improvement over the state of affairs in [14] where the blowup parameters were not specified.
2. As seen in [12],  $\theta = 0$  is the one case where the deformed orbifold is described by a theory that also has a Coulomb branch. We are certainly dealing in the fivebrane problem with a theory that has a Coulomb branch, so this fact fits nicely.
3. Finally, though it can be seen in an effective  $U(1)$  theory near the singularity, the theta angle in question is not naturally embedded in the underlying  $U(n)$  gauge theory. It is thus very natural that the underlying  $U(n)$  theory would automatically lead to  $\theta = 0$  and not  $\theta = \pi$  as the theory on the Higgs branch.

Even if it is true that the fivebrane is described by the orbifold theory deformed to  $\theta = 0$ , it is not at all clear how much computational power concerning fivebranes this will yield. The  $\theta = 0$  theory is probably strongly coupled and difficult to understand. Since this description does differ from the soluble orbifold by a marginal deformation (albeit one that violates the six-dimensional Lorentz invariance) it is plausible that it could be used as the basis for a counting of BPS states roughly along lines guessed in [38]. However, all this discussion depended upon assuming that the  $n = k = 1$  theory has a quantum Higgs vacuum.

**Extension To  $k > 1$**  We will now discuss the interactions in the case of  $k$  parallel fivebranes with  $k > 1$ . In this case, less precise statements can be made, but at least we do not need any doubtful assumptions about existence of a quantum Higgs branch.

To analyze this case, we must understand the Higgs branch of the  $U(n)$  theory with  $k$  hypermultiplets in the fundamental representation. In other words, we must understand the conformal field theory of the Higgs branch  $\mathcal{M}_{n,k}$ . A partial answer is provided by the result explained in section 3.2 that  $\mathcal{M}_{n,k}$  is birational to a symmetric product of  $n$  copies of  $\mathcal{M}_{1,k}$ . However, this result was not accompanied by a useful description of the birational transformation involved. (Somewhat more detail can be found in [37].) Ideally, one would like to understand the  $U(n)$  theory with  $k$  extra hypermultiplets as a marginal deformation of an orbifold of  $n$  copies of a  $U(1)$  theory with  $k$  charged hypermultiplets (and a free hypermultiplet, corresponding to the adjoint representation of  $U(1)$ ). This would be a more far-reaching version of the relationship between the moduli spaces  $\mathcal{M}_{n,k}$  and  $\mathcal{M}_{1,k}$ . It is not at all clear, however, that such a description should be expected.

What could one learn from such a description, if it does exist? There are many indications [40, 41, 42, 12, 43, 44] that duality and dynamics of four-dimensional gauge theories can be profitably derived from six dimensions. In particular, a large  $k$  expansion of the fivebrane theory might well lead in time to a  $1/k$  expansion of  $SU(k)$  gauge theory in four dimensions, perhaps ultimately shedding great light on QCD. The relation of the moduli space  $\mathcal{M}_{n,k}$  to a symmetric product of  $n$  copies of  $\mathcal{M}_{1,k}$  is a hint that such an expansion may exist; for  $\mathcal{M}_{1,k}$  is the Higgs branch of a theory ( $U(1)$  with  $k$  charged hypermultiplets) that does have a  $1/k$  expansion, as was emphasized in section 2.

**Compactification On  $\mathbf{T}^4$**  In [14], the compactification of the four transverse dimensions of the fivebrane on a four-torus  $\mathbf{T}^4$  was considered. It was proposed that in this case, the matrix string theory of  $k$  fivebranes is a sigma model whose target space is the moduli space of instanton number  $n$  solutions of  $U(k)$  Yang-Mills theory on  $\mathbf{T}^4$ . It is actually true [45, 46] that, at least with some restrictions on the flat metric on the  $\mathbf{T}^4$  and the Chern classes of the instanton bundle, this moduli space is a blowup of an orbifold which is a symmetric product  $S^{kn}\mathbf{T}^4$ . The idea in [14] is that interactions would be generated by the blowup (and perhaps also, in view of what has been said above, by a shift in the theta angle of the orbifold). While this proposal is perhaps plausible, it would be beyond the scope of the present paper to analyze it, since the logic of our discussion limits us to fivebranes on a transverse  $\mathbf{R}^4$ . In fact, the experience in the matrix model is that compactification brings many surprises, though perhaps in the case of fivebranes these do not affect the relationship between the light cone sigma model and the instanton moduli space.

Note that there does not appear to be a known birational relation of the moduli space of  $n U(k)$  instantons on  $\mathbf{R}^4$  to a symmetric product of  $nk$  copies of  $\mathbf{R}^4$ .<sup>7</sup> Such a relation, at

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<sup>7</sup>Though not obviously relevant to proposals about fivebrane interactions, a birational relation of  $\mathcal{M}_{n,k}$  to a symmetric product of  $n$  copies of  $\mathbf{R}^{4k}$  can be deduced using the description in section 3.2. This is a consequence of the fact that  $\mathcal{M}_{n,k}$  is birational to a symmetric product of  $n$  copies of  $\mathcal{M}_{1,k}$ ,

any rate, could apparently not commute with the action on the instanton moduli space of the global symmetry group  $SU(k)$ . This group acts because (as explained in the footnote in section 3.2) in defining the instanton moduli space, we consider two instantons to be gauge-equivalent only if they are equivalent by a gauge transformation that is the identity at infinity. After dividing by this equivalence, one has a global action of the  $U(k)$  gauge transformations at infinity on the instanton moduli space. By contrast, the group  $U(k)$  does not act in any evident fashion on the symmetric product  $S^{nk}\mathbf{R}^4$ .

There is no obvious and useful relation between instanton moduli spaces on  $\mathbf{T}^4$  and on  $\mathbf{R}^4$ . They differ because on  $\mathbf{T}^4$  one has Wilson lines that have no analog on  $\mathbf{R}^4$ , and because in defining instanton moduli space on  $\mathbf{R}^4$ , one divides only by those gauge transformations that equal the identity at infinity, a condition that has no analog on  $\mathbf{T}^4$ . A linear sigma model whose Higgs branch is a rough analog of instanton moduli space on  $\mathbf{T}^4$  would be a  $U(n) \times U(k)$  gauge theory with hypermultiplets consisting of a copy of the adjoint representation of each of the two factors together with an  $(n, k)$  hypermultiplet.

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and  $\mathcal{M}_{1,k}$  is birational to  $\mathbf{C}^{2k}$ . The latter statement can be proved by using the  $\mathbf{C}^*$  action to set one of the  $A_\alpha$ , say  $A_1$ , to 1, and then using the equation  $\sum A_\alpha B_\alpha = 0$  to eliminate  $B_1$ , leaving  $u, v$  and the other  $A$ 's and  $B$ 's to generically parametrize  $\mathcal{M}_{1,k}$ .

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